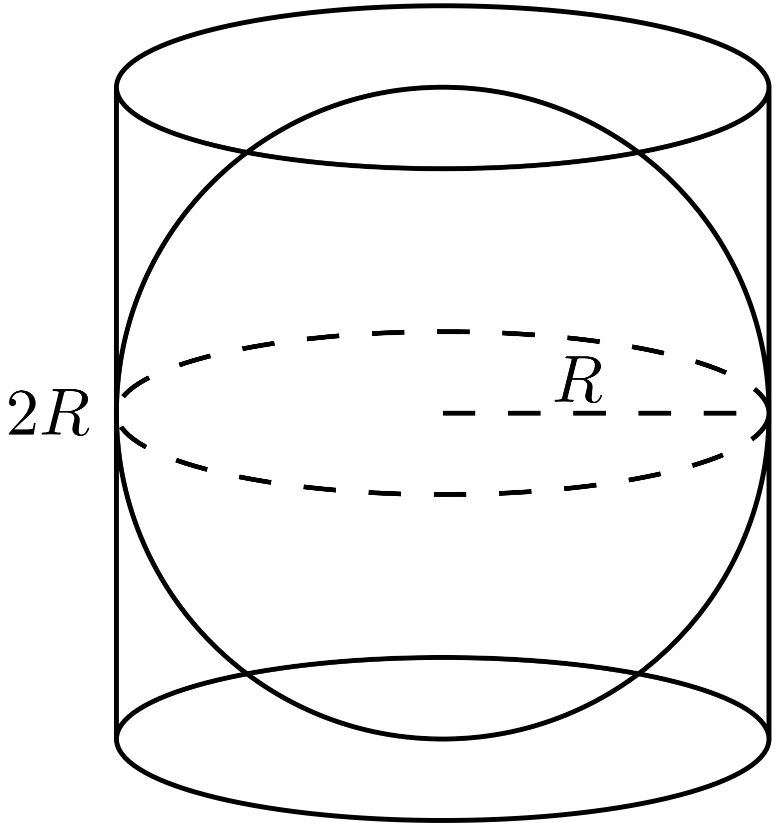
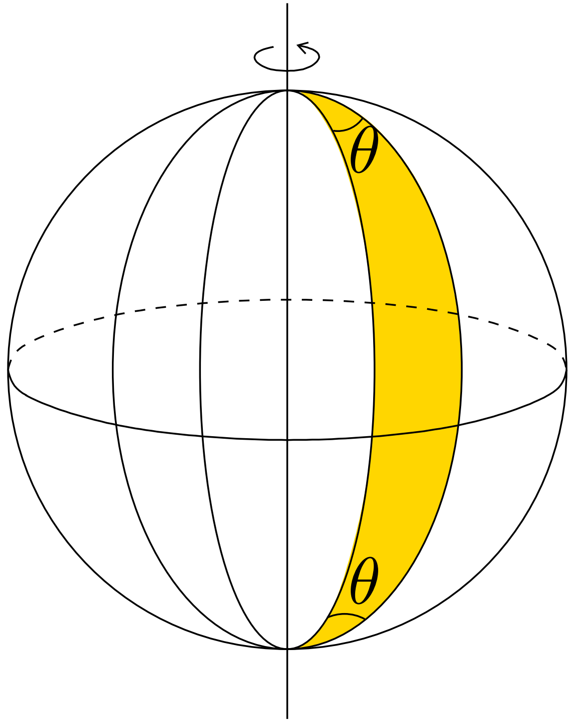
Archimedes' formula: $A = 4\pi R^2$



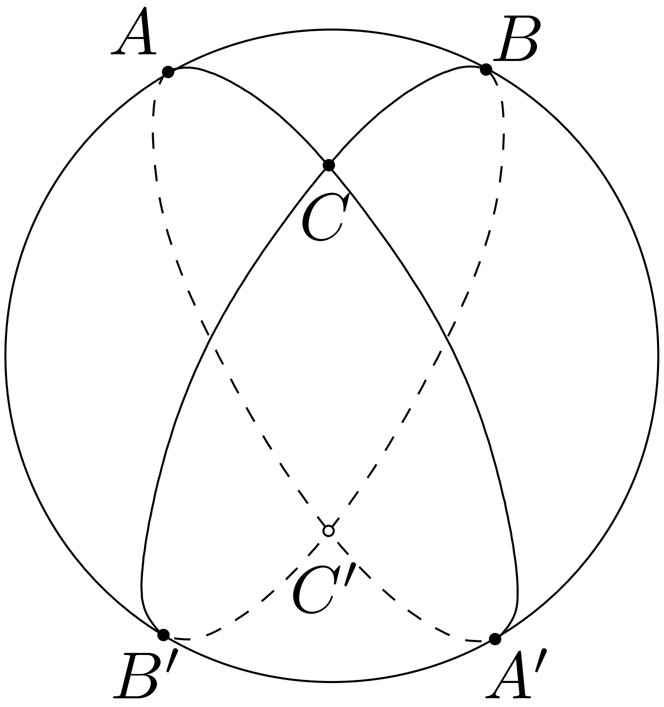
From now on, take $R = 1$.

Localization I: area of a spherical lune $= 2\theta$



Localization II (**Girard's Theorem**): The area of a spherical triangle $\triangle ABC$ on the unit sphere is:

$$\triangle = \angle A + \angle B + \angle C - \pi.$$



$$\triangle ABC + \triangle A'BC = 2\angle A \quad \triangle ABC + \triangle AB'C = 2\angle B$$
$$\triangle ABC + \triangle ABC' = 2\angle C \quad \triangle ABC' = \triangle A'B'C$$

and

$$\triangle ABC + \triangle A'BC + \triangle AB'C + \triangle A'B'C = 2\pi,$$

Summing up:

$$3\triangle ABC+\triangle A'BC+\triangle AB'C+\triangle A'B'C=2\angle A+2\angle B+2\angle C$$

Then we can decude that:

$$2\triangle ABC+2\pi=2\angle A+2\angle B+2\angle C$$

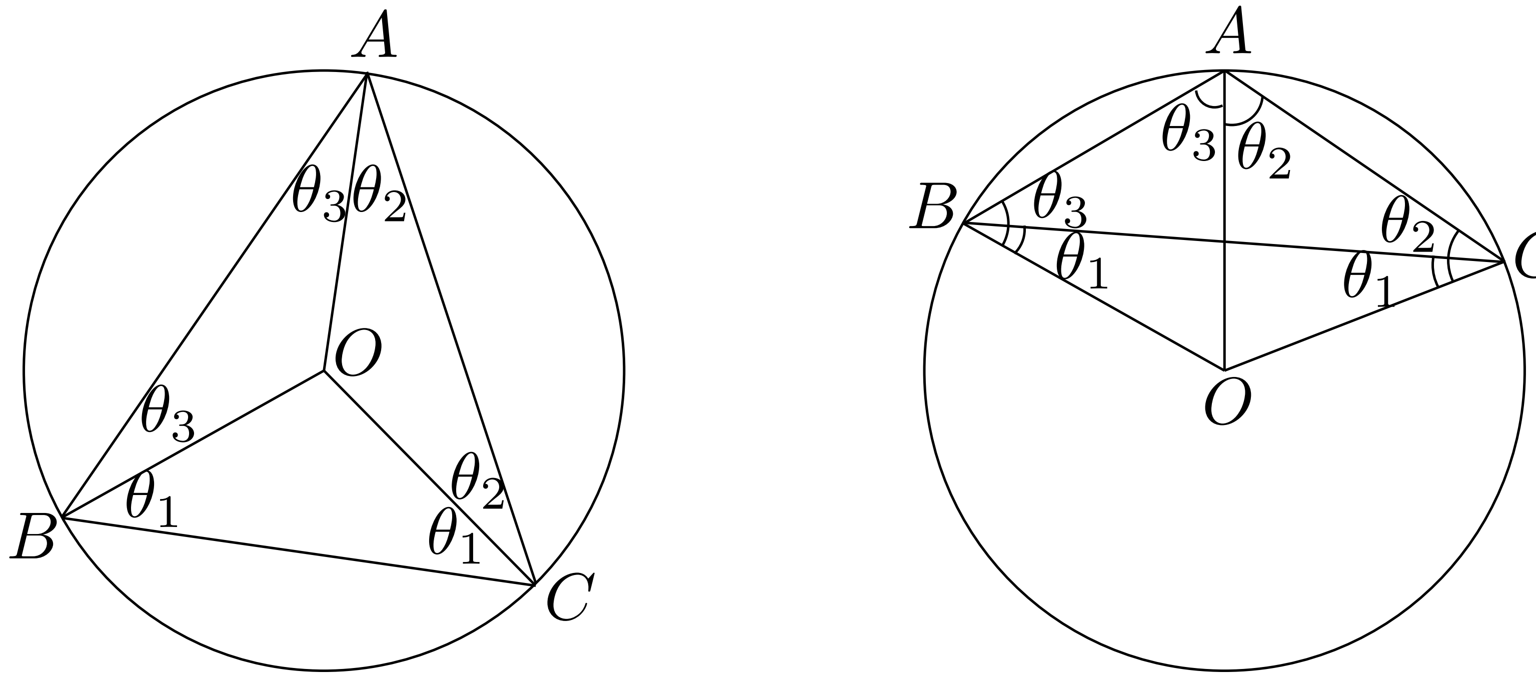
and hence:

$$\triangle ABC=\angle A+\angle B+\angle C-\pi.$$

Angle in the same segment, spherical version:

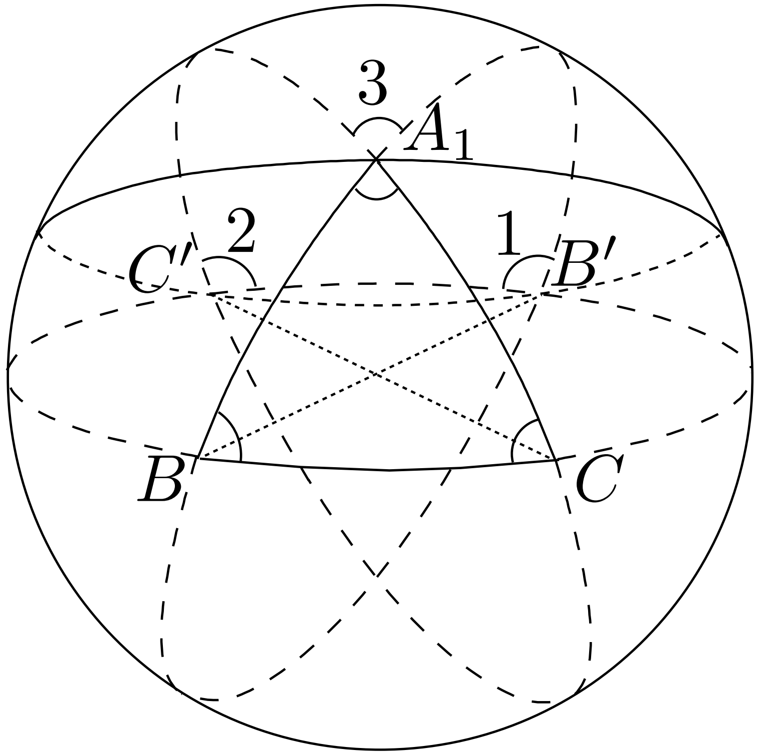
If A, A', B, C are cocirular and A, A' are lying on the same side of BC, then:

$$\angle ABC + \angle ACB - \angle A = 2\theta_1 = \angle A'BC + \angle A'CB - \angle A'$$



(i) (ii)

Lexell Theorem: Suppose that $\triangle A_1BC$ and $\triangle A_2BC$ are having the same (oriented) area. Let B', C' denote the antipodal points of B, C. Then B', C', A_1, A_2 are cocircular.



Consider the inner angles of $\triangle A_1B'C'$.

It is easy to see:

$$\angle 1 = \pi - B,\ \angle 2 = \pi - C,\ \angle 3 = A_1.$$

Then:

$$\angle 1 + \angle 2 - \angle 3 = \pi - \triangle A_1BC$$

Similarly, we have:

$$\angle 1' + \angle 2' - \angle 3' = \pi - \triangle A_2BC$$

So, if $\triangle A_1BC = \angle A_2BC$, we have:

$$\angle 1 + \angle 2 - \angle 3 = \angle 1' + \angle 2' - \angle 3',$$

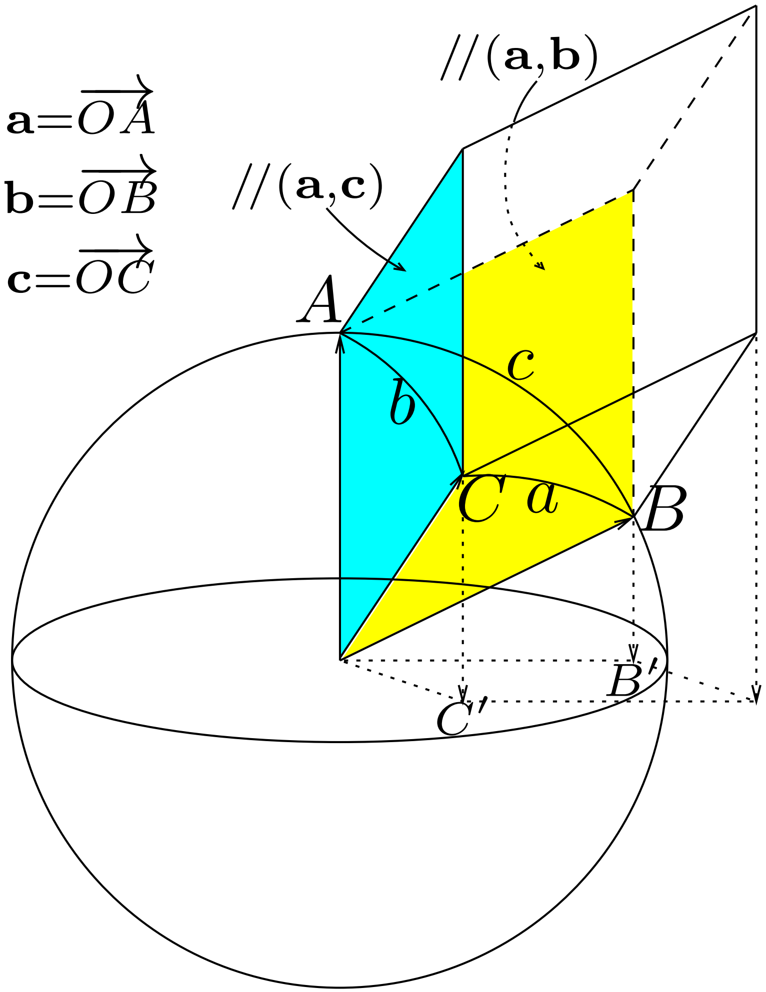
and hence the four points B', C', A_1, A_2 will be cocircular.

Vector algebra and Spherical trigonometry

$\mathbf{a}=\overrightarrow{OA}$

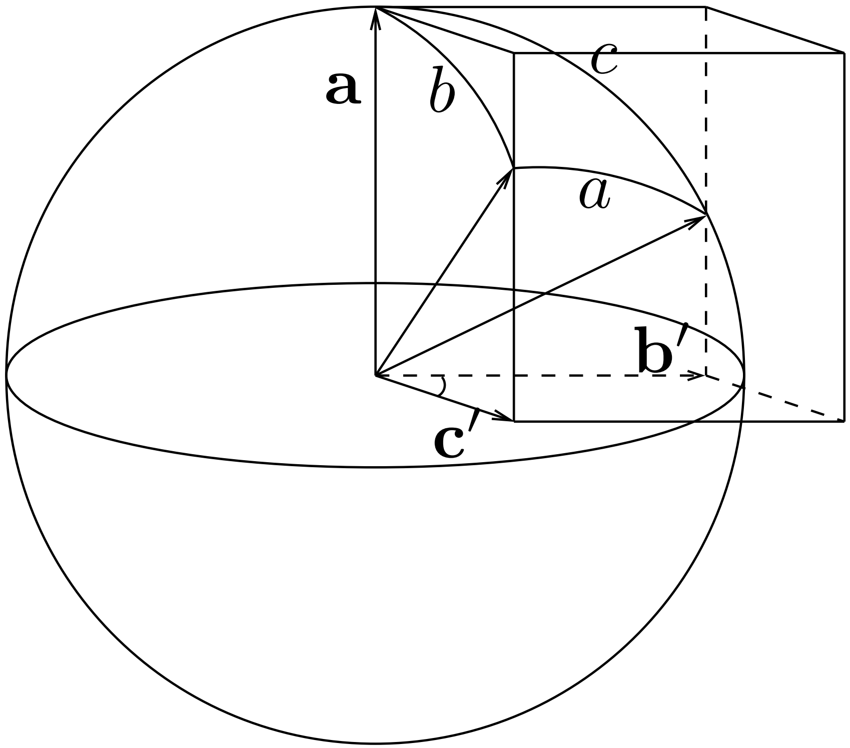
$\mathbf{b}=\overrightarrow{OB}$

$\mathbf{c}=\overrightarrow{OC}$



$\mathbf{b}'=\overrightarrow{OB'}=\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\mathbf{a},$

$\mathbf{c}'=\overrightarrow{OC'}=\mathbf{c}-(\mathbf{c}\cdot\mathbf{a})\mathbf{a}$



$$
\begin{aligned}
\operatorname{vol}/\!\!/(\mathbf{a},\mathbf{b},\mathbf{c}) &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}'\times\mathbf{c}') \\
&= \operatorname{area}/\!\!/(\mathbf{b}',\mathbf{c}') = |\mathbf{b}'|\,|\mathbf{c}'|\sin A \\
&= \sin c \sin b \sin A
\end{aligned}
$$

So we have:

$$\frac{\sin A}{\sin a} = \frac{\operatorname{vol}/\!/(\mathbf{a},\mathbf{b},\mathbf{c})}{\sin a \sin b \sin c} \qquad \left(= \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \right)$$

On the other hand,

$$\begin{aligned}
\mathbf{b}' \cdot \mathbf{c}' &= |\mathbf{b}'|\,|\mathbf{c}'| \cos A = \sin c \sin b \cos A \\
&= \big(\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{a}\big) \cdot \big(\mathbf{c} - (\mathbf{c}\cdot\mathbf{a})\mathbf{a}\big) \\
&= \mathbf{b}\cdot\mathbf{c} - (\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{a}) - (\mathbf{c}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{a}) + (\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{a}) \\
&= \cos a - \cos b \cos c.
\end{aligned}$$

1. **Spherical Sine Laws**:

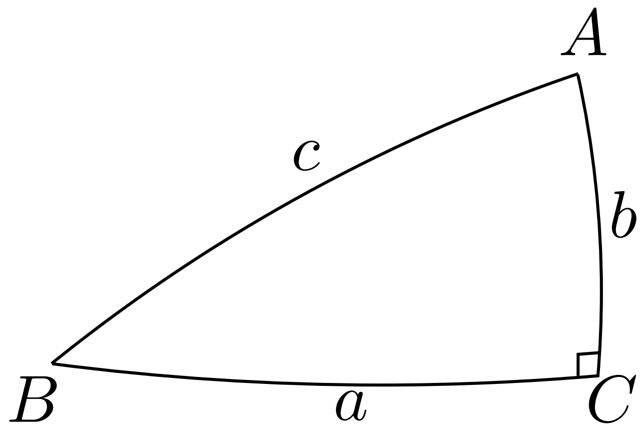
$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{D}{\sin a \sin b \sin c},$$

$$D = \mathrm{vol}\,/\!/(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0.$$

2. **Spherical Cosine Laws**:

$$\sin b \sin c \cos A = \cos a - \cos b \cos c,$$
$$\sin c \sin a \cos B = \cos b - \cos c \cos a,$$
$$\sin a \sin b \cos C = \cos c - \cos a \cos b.$$

3. Special case of right-angle triangle:



$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{1}{\sin c}$$

$$\Rightarrow \sin A = \frac{\sin a}{\sin c}, \quad \sin B = \frac{\sin b}{\sin c}.$$

$$\sin a \sin b \cos C = \cos c - \cos a \cos b$$

$$\Rightarrow \cos c = \cos a \cos b.$$

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$
$$= \frac{\cos a(1 - \cos^2 b)}{\sin b \sin c} = \frac{\cos a \sin b}{\sin c}$$
$$= \frac{\cos c \tan b}{\sin c} = \frac{\tan b}{\tan c}.$$

Similarly, $\cos B = \dfrac{\tan a}{\tan c}$.

$$\tan A = \frac{\sin a}{\sin c} \cdot \frac{\sin c}{\cos c \tan b} = \frac{\tan a}{\sin b}.$$

Similarly, $\tan B = \dfrac{\tan b}{\sin a}$.

Area formula in terms of side-lengths a, b, c:

$$\begin{aligned}\triangle &= \angle A + \angle B + \angle C - \pi \\ &= \cos^{-1}\left\{\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right\} \\ &\quad + \cos^{-1}\left\{\frac{\cos b - \cos c \cos a}{\sin c \sin a}\right\} \\ &\quad + \cos^{-1}\left\{\frac{\cos c - \cos a \cos b}{\sin a \sin b}\right\} - \pi\end{aligned}$$

... *not convenient to use directly.*

Theorem (Euler, 1781):

$$\tan\frac{\triangle}{2} = \frac{D}{u},$$

where:

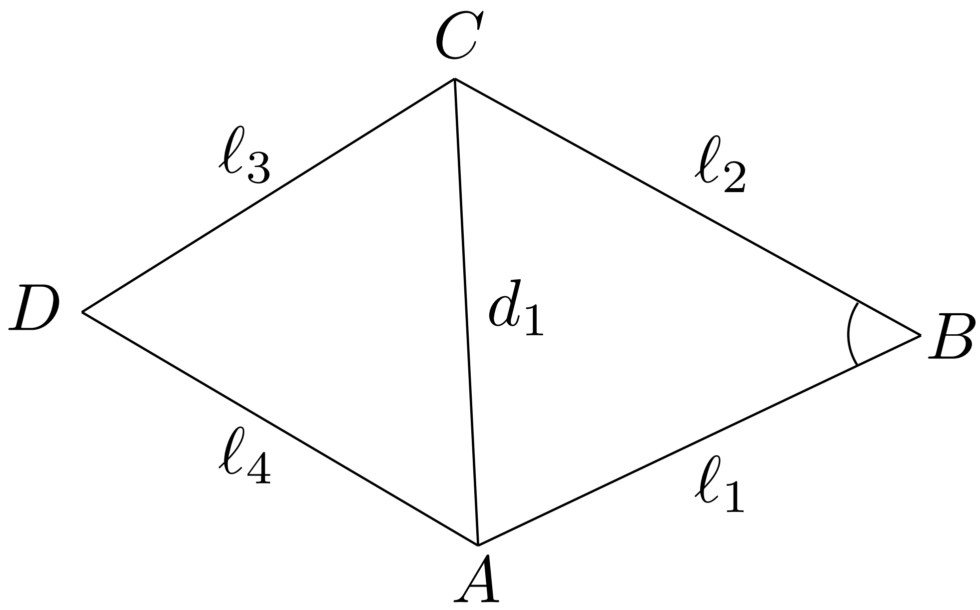
$$u = 1 + \cos a + \cos b + \cos c,$$
$$D = \{1 + 2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c\}^{\frac{1}{2}}.$$

Corollary: Let $x = 1 + \cos c$. Then:

$$\frac{d\triangle}{dx} = \frac{\cos a + \cos b - x}{xD}, \quad \frac{d\triangle}{dC} = \frac{x - \cos a - \cos b}{x}.$$

Spherical Quadrilaterals

A spherical quadrilateral can be determined by its 4 side-lengths and 1 diagonal length. So there should be a relation among the six length parameters $(\ell_1, \ell_2, \ell_3, \ell_4; d_1, d_2)$.



The determinant formula will capture the relation:

$$\det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 \det(\mathbf{a}, \mathbf{c}, \mathbf{d})^2 = [\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) \det(\mathbf{a}, \mathbf{c}, \mathbf{d})]^2.$$

Another one:

$$\det(\mathbf{a},\mathbf{b},\mathbf{d})^2\det(\mathbf{b},\mathbf{c},\mathbf{d})^2=[\det(\mathbf{a},\mathbf{b},\mathbf{d})\det(\mathbf{b},\mathbf{c},\mathbf{d})]^2.$$

The relation is rather complicated in general, but in the special case of $\ell_1=\ell_3=a$, $\ell_2=\ell_4=b$ (equal opposite sides, parallelogram-like), the relation simplifies greatly to:

$$(1+\cos d_1)(1+\cos d_2)=(\cos a+\cos b)^2.$$

If in additional we also have $d_1=d_2$ (equal diagonal lengths, rectangle-like), we have:

$$1+\cos d=\cos a+\cos b.$$

Area formula of a spherical "rectangle":

The area of $\square(a,b)$, the spherical "rectangle" of sides (a,b,a,b) and $d_1 = d_2$, is given by:

$$\square(a,b) = 4\tan^{-1}\left\{\frac{(1-\cos a)(1-\cos b)}{2(\cos a + \cos b)}\right\}^{\frac{1}{2}}.$$

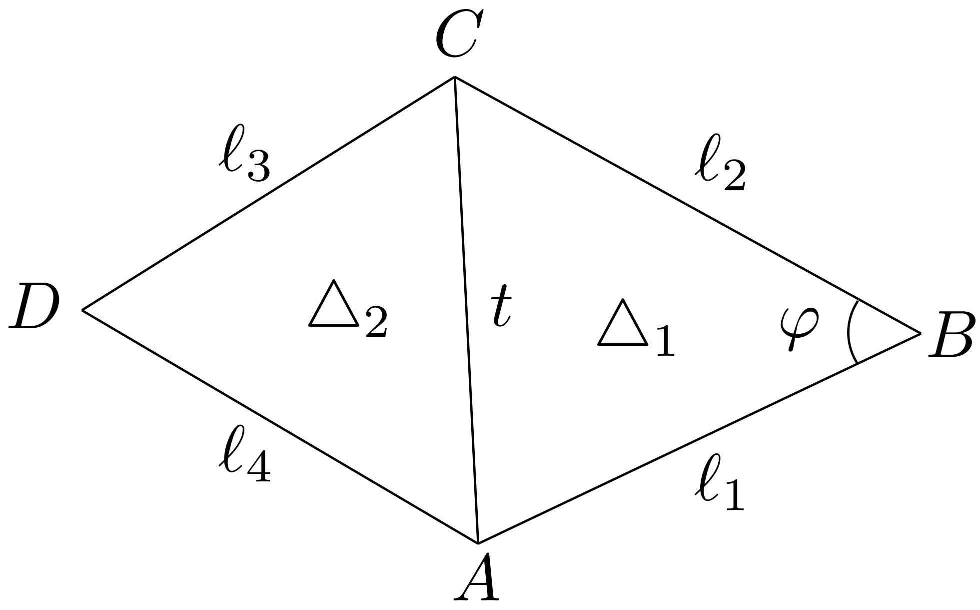
Proof: The rectangle can be cut into two triangles with sides (a,b,d) with $1+\cos d = \cos a + \cos b$. So:

$$\begin{aligned} u &= \cos a + \cos b + \cos d + 1 = 2(\cos a + \cos b), \\ D^2 &= 1 + 2\cos a \cos b \cos d - \cos^2 a - \cos^2 b - \cos^2 d \\ &= 2(\cos a + \cos b)(1-\cos a)(1-\cos b). \end{aligned}$$

Then result follows by substitution in $\square(a,b) = 4\tan^{-1}\frac{D}{u}$.

Shearing deformation of quadrilaterals

Consider a (convex) quadrilateral $\square ABCD$ with varying diagonal length AC:



Its area is obviously:

$$\square = \triangle_1 + \triangle_2 = 2\tan^{-1}\frac{D_1}{u_1} + 2\tan^{-1}\frac{D_2}{u_2}.$$

Let $x = 1 + \cos t$, $c_i = \cos \ell_i$. Then:

$$\frac{d\Box}{dx} = \frac{c_1 + c_2 - x}{xD_1} + \frac{c_3 + c_4 - x}{xD_2}.$$

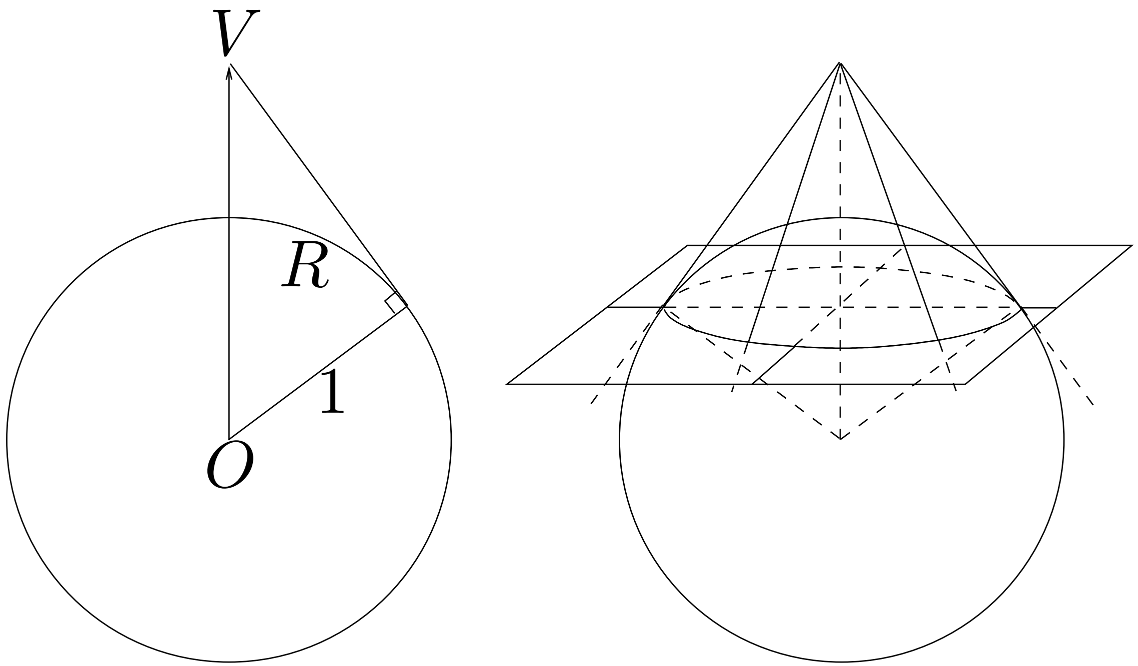
What is the geometric meaning of having maximal area:

$$\frac{d\Box}{dx} = 0\,?$$

Answer: The circumcenters of $\triangle_1$, $\triangle_2$ coincides, i.e. the quadrilateral is cocircular.

How do we express the circumcenter center of a spherical triangle in formula?

Formula for circumcenter:



Spherical circumcircle coincides with Euclidean circumcircle.

$$\overrightarrow{OV} = \mathbf{v} = \frac{1}{D}(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$$

$$\begin{aligned}
\sec^2 R = |\mathbf{v}|^2 &= \mathbf{v}\cdot\mathbf{v} \\
&= \frac{1}{D^2}(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})\cdot \\
&\qquad (\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}) \\
&= \frac{1}{D^2}\{3-\sum\cos^2 a+2\sum\cos a\cos b \\
&\qquad -2\sum\cos a\}
\end{aligned}$$

$$\Rightarrow\ \tan^2 R = \frac{2}{D^2}(1-\cos a)(1-\cos b)(1-\cos c)$$

Theorem: Let **a**, **b**, **c**, **d** be the vertices of a quadrilateral, and let $\overrightarrow{OV}_1$ and $\overrightarrow{OV}_2$ be given by:

$$\overrightarrow{OV_1} = \frac{1}{\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}} \{\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}\},$$
$$\overrightarrow{OV_2} = \frac{1}{\mathbf{a} \times \mathbf{c} \cdot \mathbf{d}} \{\mathbf{a} \times \mathbf{c} + \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{a}\}.$$

Then:

$$\overrightarrow{V_1 V_2} = \frac{d\Box}{dt} \frac{\mathbf{a} \times \mathbf{c}}{|\mathbf{a} \times \mathbf{c}|}, \quad \frac{d\Box}{dB} = \overrightarrow{V_2 V_1} \cdot \mathbf{b}.$$

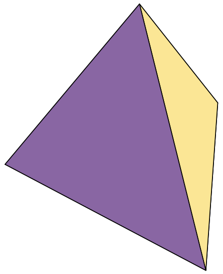
Some additional technical details:

Sublemma 1: The circumference of a convex spherical polygon is less than 2π.

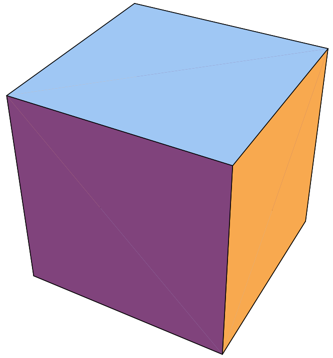
Sublemma 2: If $\sum \ell_i < 2\pi$, the spherical polygon is contained in an open hemisphere.

Corollay: Among the spherical polygons with a set of given side lengths in cyclic order and total length less than 2π, the area maximizing one is exactly the cocircular one.

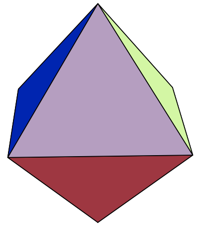
Applications: regular polyhedrons



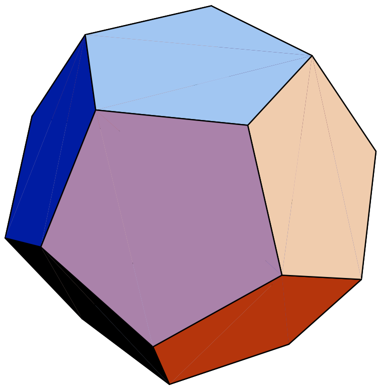
Tetrahedron



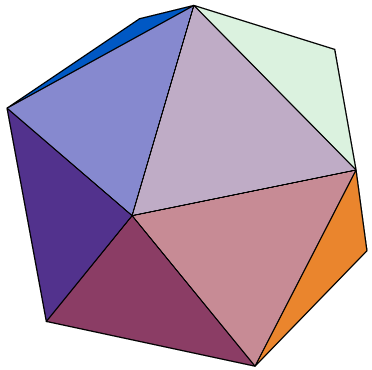
Cube



Octahedron

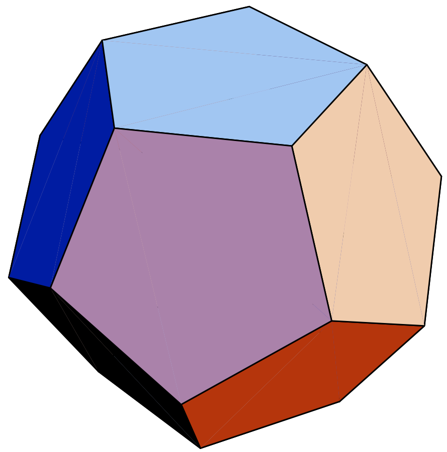


Dodecahedron



Icosahedron

Dodecahedron: has 12 regular pentagonal faces



To find its volume, we use cone decomposition.

A natural choice of the common vertex of the cones will be the center of either the insphere or the circumscribing sphere.

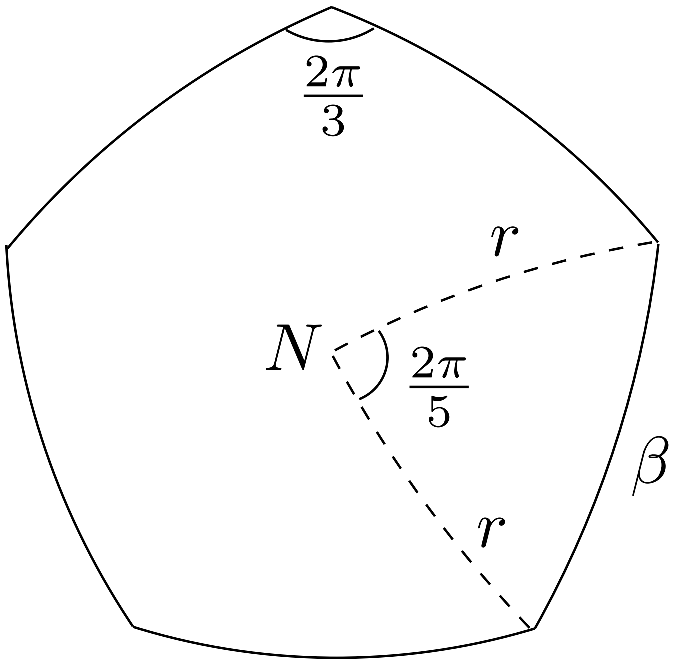
The computation using insphere turns out to be more involved.

Using circumscribing sphere:

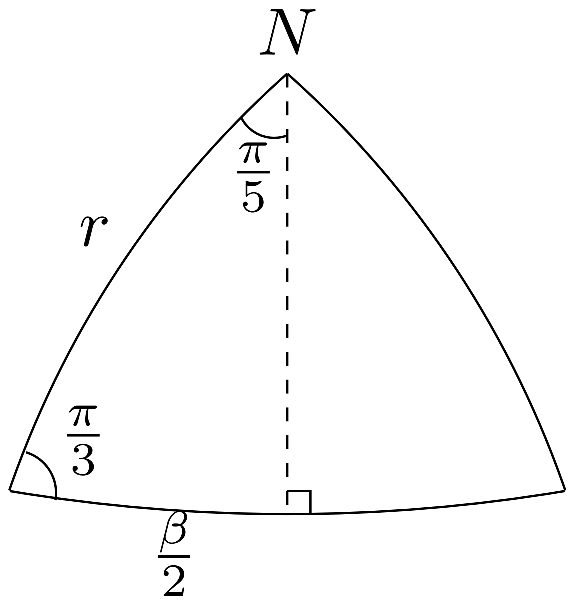
Set the radius of circumscribing sphere to be 1.

Using radial projection from the center of the circumscribing sphere, we obtain a decomposition of the sphere into 12 identical regular spherical pentagons.

As each vertex is shared by 3 pentagons, the inner angle of each spherical pentagon should be $\frac{2\pi}{3}$.



Let N denote the circumcenter of the pentagon, and let r denote its circumraidus. Set β to be the side length. (All on the sphere.)

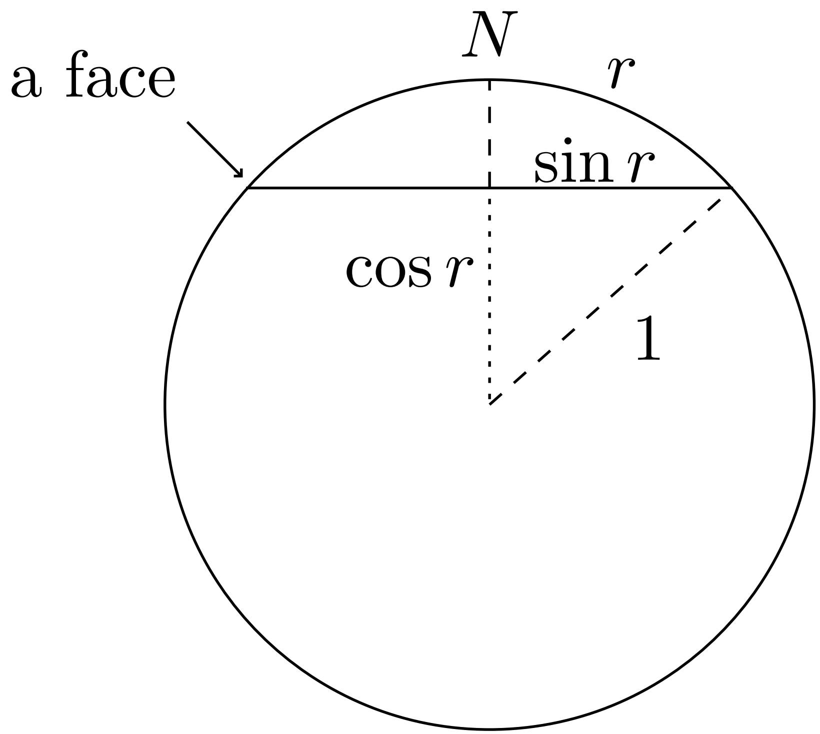


$$\sin\frac{\pi}{5} = \frac{\sin\frac{\beta}{2}}{\sin r}, \quad \cos\frac{\pi}{3} = \frac{\tan\frac{\beta}{2}}{\tan r}$$

From these we can solve that:

$$\cos\beta = \frac{\sqrt{5}}{3}, \quad \sin^2 r = \frac{4(3-\sqrt{5})}{3(5-\sqrt{5})}$$

Looking at the original dodecahedron from the side:



The area of the pentagonal face is:

$$5 \cdot \frac{1}{2} \sin^2 r \sin \frac{2\pi}{5},$$

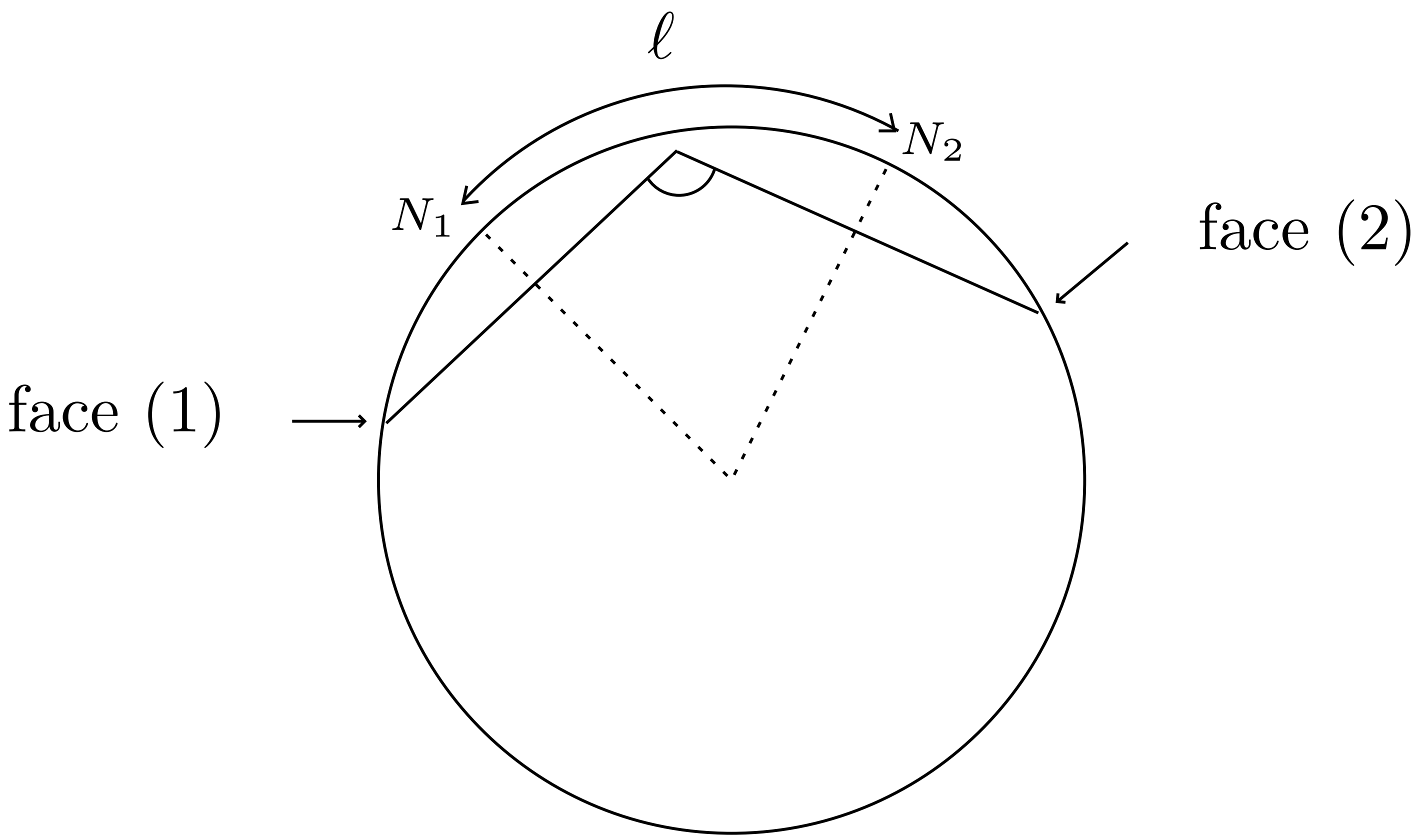
and hence the volume of the cone formed:

$$\frac{1}{3} \cdot 5 \cdot \frac{1}{2} \sin^2 r \sin \frac{2\pi}{5} \cdot \cos r$$

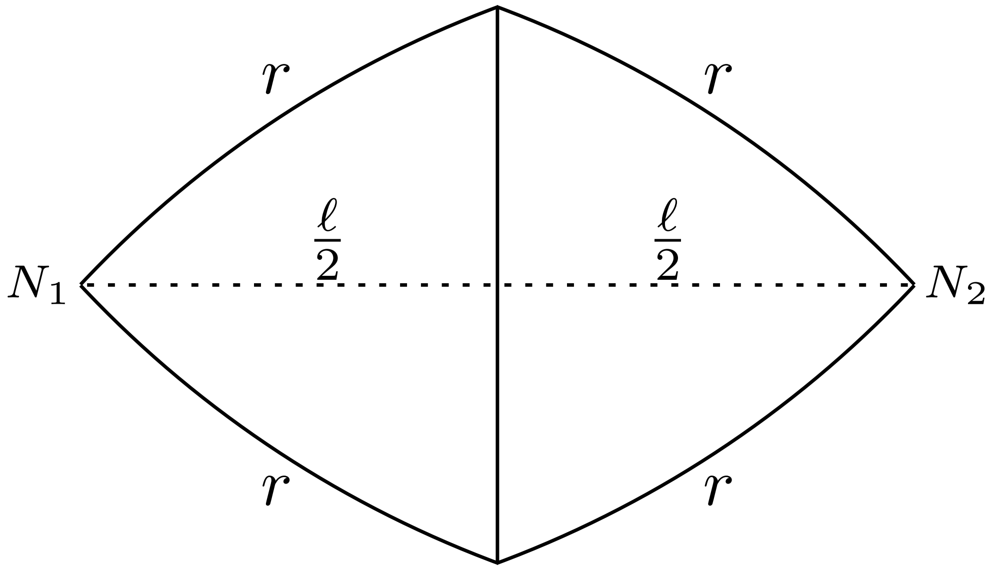
Volume of the dodecahedron:

$$\frac{2}{3}\sqrt{\frac{5}{3}}(1+\sqrt{5}) = 2.785163863\ldots$$

Computing dihedral angle:



The required dihedral angle should be $\pi - \ell$, where ℓ is the spherical distance between the two circumcenters.



$$\sin^2 r \cos\frac{2\pi}{3} = \cos\ell - \cos^2 r$$

$$\Rightarrow \cos\ell = 1 - \frac{3}{2}\sin^2 r = \frac{1}{\sqrt{5}}$$

$$\Rightarrow \pi - \ell = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \approx 116.6^\circ$$